

TORSIONFREE CRYSTALLOGRAPHIC GROUPS WITH INDECOMPOSABLE HOLONOMY GROUP

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ABSTRACT. Let K be a principal ideal domain, G a finite group, and M a KG -module which as K -module is free of finite rank, and on which G acts faithfully. A *generalized crystallographic group* (introduced by the authors in volume 5 of this Journal) is a group \mathfrak{C} which has a normal subgroup isomorphic to M with quotient G , such that conjugation in \mathfrak{C} gives the same action of G on M that we started with. (When $K = \mathbb{Z}$, these are just the classical crystallographic groups.) The K -free rank of M is said to be the *dimension* of \mathfrak{C} , the *holonomy group* of \mathfrak{C} is G , and \mathfrak{C} is called *indecomposable* if M is an indecomposable KG -module.

Let K be either \mathbb{Z} , or its localization $\mathbb{Z}_{(p)}$ at the prime p , or the ring \mathbb{Z}_p of p -adic integers, and consider indecomposable torsionfree generalized crystallographic groups whose holonomy group is noncyclic of order p^2 . In Theorem 2, we prove that (for any given p) the dimensions of these groups are not bounded.

For $K = \mathbb{Z}$, we show in Theorem 3 that there are infinitely many non-isomorphic indecomposable torsionfree crystallographic groups with holonomy group the alternating group of degree 4. In Theorem 1, we look at a cyclic G whose order $|G|$ satisfies the following condition: for all prime divisors p of $|G|$, p^2 also divides $|G|$, and for at least one p , even p^3 does. We prove that then every product of $|G|$ with a positive integer coprime to it occurs as the dimension of some indecomposable torsionfree crystallographic group with holonomy group G .

1. INTRODUCTION

Zassenhaus developed algebraic methods in [11] for studying the classical crystallographic groups and he pointed out the close connection between the theory of crystallographic groups and the theory of integral representations of finite groups. A historical overview and the present state of the theory of crystallographic groups as well as its connections to other branches of mathematics are described in [9, 10].

It was shown in [5, 7] that, in general, the classification of the crystallographic groups is a problem of wild type, in the sense that it is related to the classical unsolvable problem of describing the canonical forms of pairs of linear operators acting on finite dimensional vector spaces. One may therefore focus attention on certain special classes of crystallographic groups, for example, on groups whose

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translation group affords an irreducible (or indecomposable) integral representation of the holonomy group. In this direction, Hiss and Szczepański in [6] proved that torsionfree crystallographic groups with irreducible holonomy group do not exist. On the other hand, Kopcha and Rudko in [7] showed that the problem of describing torsionfree crystallographic groups with indecomposable cyclic holonomy group of order p^n , $n \geq 5$, is still of wild type.

The *generalized crystallographic groups* which we introduced in [3] may be defined as follows. Let K be a principal ideal domain, G a finite group, and M a KG -module which as K -module is free of finite rank, and on which G acts faithfully. A generalized crystallographic group is a group \mathfrak{C} which has a normal subgroup isomorphic to M with quotient G , such that conjugation in \mathfrak{C} gives the same action of G on M that we started with, and that the extension in question does not split. The K -free rank of M is said to be the *dimension* of \mathfrak{C} , and the *holonomy group* of \mathfrak{C} is G . (In the special case of $K = \mathbb{Z}$, this matches one of the usual descriptions of crystallographic groups; for emphasis, we sometimes refer to those as *classical* crystallographic groups.)

In [3], we looked at indecomposable generalized crystallographic groups when K is either \mathbb{Z} , or its localization $\mathbb{Z}_{(p)}$ at the prime p , or the ring \mathbb{Z}_p of p -adic integers, and either G is a cyclic p -group or $p = 2$ and G is non-cyclic of order 4. Retaining this restriction on the choice of K but allowing p to be arbitrary, we consider here indecomposable torsionfree generalized crystallographic groups whose holonomy group is noncyclic of order p^2 . In Theorem 2, we prove that (for any given p) the dimensions of such groups are not bounded.

For the classical case ($K = \mathbb{Z}$), we show in Theorem 3 that there are infinitely many non-isomorphic indecomposable torsionfree crystallographic groups with holonomy group the alternating group of degree 4. In Theorem 1, we look at a cyclic G whose order $|G|$ satisfies the following condition: for all prime divisors p of $|G|$, p^2 also divides $|G|$, and for at least one p , even p^3 does. We prove that then every product of $|G|$ with a positive integer coprime to it occurs as the dimension of some indecomposable torsionfree crystallographic group with holonomy group G .

2. THE MAIN RESULTS

For the formal statement of our results, we need some terminology.

Let K be a principal ideal domain, F be a field containing K and let G be a finite group. Let M be a K -free KG -module, with a finite K -basis affording a faithful representation Γ of G by matrices over K ; further, let FM be the F -space spanned by this K -basis of M , so M becomes a full lattice in FM . Let $\widehat{M} = FM^+/M^+$ be the quotient group of the additive group FM^+ of the linear space FM by the additive group M^+ of the module M . Then FM is an FG -module and \widehat{M} is a KG -module with operations:

$$g(\alpha m) = \alpha g(m); \quad g(x + M) = g(x) + M,$$

where $g \in G$, $\alpha \in F$, $m \in M$, $x \in FM$.

Let $T : G \rightarrow \widehat{M}$ be a 1-cocycle of G with values in \widehat{M} . Elements of \widehat{M} being cosets in FM^+ modulo M^+ , we consider each value $T(g)$ of T a subset of FM , and define the group

$$\mathfrak{Crns}(G; M; T) = \{ (g, x) \mid g \in G, x \in T(g) \}$$

with the operation

$$(g, x)(g', x') = (gg', gx' + x),$$

where $g, g' \in G, x \in T(g), x' \in T(g')$.

The K -free rank of M will be called the K -dimension of the group $\mathfrak{Crns}(G; M; T)$. When T is not cohomologous to 0, the group $\mathfrak{Crns}(G; M; T)$ is called *indecomposable* if M is an indecomposable KG -module.

We note that if $K = \mathbb{Z}$ and $F = \mathbb{R}$, then the abstract group $\mathfrak{Crns}(G; M; T)$ is isomorphic to a classical crystallographic group.

Let $C^1(G, \widehat{M})$ and $B^1(G, \widehat{M})$ be the group of the 1-cocycles and group of the 1-coboundaries of G with values in the module \widehat{M} , respectively, so that $H^1(G, \widehat{M}) = C^1(G, \widehat{M})/B^1(G, \widehat{M})$. The group $\mathfrak{Crns}(G; M; T)$ is an extension of M^+ by the group G . The group $\mathfrak{Crns}(G; M; T)$ is torsionfree if and only if for each prime order subgroup H of G the restriction $T|_H$ is not a coboundary.

Using results from [1,2,8] we prove the following two theorems.

Theorem 1. *Let G be a cyclic group of order $|G| = p_1^{n_1} \cdots p_s^{n_s}$, where p_1, \dots, p_s are pairwise distinct primes ($n_1 \geq 3$ and if $s \geq 2$ then $n_2 \geq 2, \dots, n_s \geq 2$), let m be a natural number which is coprime to $|G|$ and put $d = m \cdot |G|$. Then there exists a torsionfree indecomposable classical crystallographic group of dimension d with holonomy group isomorphic to G .*

Theorem 2. *Let K be either \mathbb{Z} , $\mathbb{Z}_{(p)}$, or \mathbb{Z}_p , and let $G \cong C_p \times C_p$. Then the K -dimensions of the indecomposable torsionfree groups $\mathfrak{Crns}(G; M; T)$ are unbounded.*

In [3] we gave a complete description of the indecomposable torsionfree crystallographic groups whose holonomy group is $C_2 \times C_2$. Moreover, we proved that there exist at least $2p-3$ torsionfree crystallographic groups having cyclic indecomposable holonomy group of order p^2 . Note that the holonomy group of an indecomposable torsionfree crystallographic group can never have prime order. Therefore the following result is also interesting.

Theorem 3. *There exist infinitely many non-isomorphic indecomposable torsionfree classical crystallographic groups with holonomy group isomorphic to the alternating group A_4 of degree 4.*

3. PRELIMINARY RESULTS AND THE PROOF OF THEOREM 1

Let $K = \mathbb{Z}$, $\mathbb{Z}_{(p)}$ or \mathbb{Z}_p as above, $H_{p^n} = \langle a \mid a^{p^n} = 1 \rangle$ be a cyclic group of order p^n ($n \geq 2$), ξ_s be a p^s th primitive root of unity, $\xi_s^p = \xi_{s-1}$ ($s \geq 1, \xi_0 = 1$). Define

ordered bases B_i for the free K -modules $\mathfrak{R}_i = K[\xi_i]$ by setting

$$\begin{aligned} B_1 &= \{1, \xi_1, \dots, \xi_1^{p-2}\}, \\ B_2 &= \{1, \xi_1, \dots, \xi_1^{p-2}, \xi_2, \xi_2 \xi_1, \dots, \xi_2^{p-1} \xi_1^{p-2}\}, \end{aligned}$$

and in general (for $i > 1$)

$$B_i = B_{i-1} \cup \xi_i B_{i-1} \cup \xi_i^2 B_{i-1} \cup \dots \cup \xi_i^{p-1} B_{i-1},$$

ordered as indicated. Obviously, $|B_i| = \phi(p^i)$ (where ϕ is the Euler function). It is easy to check that each \mathfrak{R}_i ($i \leq n$) is a KH_{p^n} -module with action defined by

$$a(\alpha) = \xi_i \cdot \alpha, \quad (\alpha \in \mathfrak{R}_i). \quad (1)$$

Keep it in mind that \mathfrak{R}_i is only a K -submodule of \mathfrak{R}_{i+1} , not a KH_{p^n} -submodule. We define by $\tilde{\xi}_i$ the matrix of multiplication by ξ_i with respect to the K -basis B_i of the ring \mathfrak{R}_i ($i \geq 1$, $\mathfrak{R}_0 = K$). Note that

$$\tilde{\xi}_i^p = E_p \otimes \tilde{\xi}_{i-1}, \quad (i > 1)$$

where E_p is the identity matrix of degree p and \otimes is the Kronecker product of matrices.

Let δ_i be the matrix representation of H_{p^n} with respect to the K -basis B_i of the KH_{p^n} -module \mathfrak{R}_i . From (1) it follows that

$$\delta_i(a) = \tilde{\xi}_i \quad (i \geq 0, \quad \tilde{\xi}_0 = 1)$$

and $\delta_0, \dots, \delta_n$ are irreducible K -representations of H_{p^n} .

We introduce the following notation. Let $0 \leq i \leq j \leq n$. For each $\alpha \in \mathfrak{R}_i$ we denote by $\langle \alpha \rangle_j^i$ the matrix with $\phi(p^i)$ rows and $\phi(p^j)$ columns in which all columns are zero except the last which consists of the coordinates of $\alpha \in \mathfrak{R}_i$ in the basis B_i .

Now let $0 \leq i < j \leq n$. Thus

$$\begin{aligned} \tilde{\xi}_i \cdot \langle \alpha \rangle_j^i &= \langle \xi_i \alpha \rangle_j^i; \\ \langle \alpha \rangle_j^i &= (\langle 0 \rangle_{j-1}^i, \dots, \langle 0 \rangle_{j-1}^i, \langle \alpha \rangle_{j-1}^i); \\ \langle \alpha \rangle_j^i \cdot \tilde{\xi}_j^k &= (\langle \alpha_1(k) \rangle_{j-1}^i, \dots, \langle \alpha_{p-1}(k) \rangle_{j-1}^i, \langle \alpha_p(k) \rangle_{j-1}^i), \end{aligned} \quad (2)$$

where $0 \leq k < p$, $0 \leq i < j \leq n$, $\alpha_{p-k}(k) = \alpha$ and $\alpha_s(k) = 0$ for $s \neq p - k$. The matrix $\langle \alpha \rangle_j^i$ defines an extension of the KH_{p^n} -module \mathfrak{R}_i by the KH_{p^n} -module \mathfrak{R}_j in which the following K -representation of the group H_{p^n} is realized:

$$a \rightarrow \begin{pmatrix} \tilde{\xi}_i & \langle \alpha \rangle_j^i \\ 0 & \tilde{\xi}_j \end{pmatrix}. \quad (3)$$

If $\alpha \equiv 0 \pmod{p\mathfrak{R}_i}$, then the K -representation (3) of H_{p^n} is completely reducible and the corresponding extension of modules is split, i.e.

$$p \cdot \text{Ext}_{KH_{p^n}}(\mathfrak{R}_j, \mathfrak{R}_i) = 0 \quad (i > j). \quad (4)$$

Let m be a natural number and let A be an $m \times m$ matrix over K . Consider the K -representations of the cyclic group $H_{p^n} = \langle a \mid a^{p^n} = 1 \rangle$, with $n > 2$, defined by:

$$\Delta_1 = E_m \otimes \delta_0 + E_m \otimes \delta_1 : \quad a \rightarrow \begin{pmatrix} E_m & 0 \\ 0 & E_m \otimes \tilde{\xi}_1 \end{pmatrix};$$

$$\Delta_2 = E_m \otimes \delta_2 + \cdots + E_m \otimes \delta_n : \quad a \rightarrow \begin{pmatrix} E_m \otimes \tilde{\xi}_2 & & 0 \\ & \ddots & \\ 0 & & E_m \otimes \tilde{\xi}_n \end{pmatrix};$$

$$\Gamma_{p,A}^{(m)} = \begin{pmatrix} \Delta_1 & U \\ 0 & \Delta_2 \end{pmatrix} : \quad a \rightarrow \begin{pmatrix} \Delta_1(a) & U(a) \\ 0 & \Delta_2(a) \end{pmatrix},$$

where the matrix

$$U(a) = \begin{pmatrix} A \otimes \langle 1 \rangle_2^0 & E_m \otimes \langle 1 \rangle_3^0 & \cdots & E_m \otimes \langle 1 \rangle_n^0 \\ E_m \otimes \langle 1 \rangle_2^1 & E_m \otimes \langle 1 \rangle_3^1 & \cdots & E_m \otimes \langle 1 \rangle_n^1 \end{pmatrix}$$

is called the *intertwining* matrix.

For $n = 2$ we define the following K -representation of $H_{p^2} = \langle a \mid a^{p^2} = 1 \rangle$:

$$\Gamma_p^{(1)} : \quad a \rightarrow \begin{pmatrix} 1 & 0 & \langle 1 \rangle_2^0 \\ & \tilde{\xi}_1 & \langle 1 \rangle_2^1 \\ 0 & & \tilde{\xi}_2 \end{pmatrix}. \quad (5)$$

Lemma 1. *Let J_m be the lower triangular Jordan block of degree m with ones in the main diagonal. Then $\Gamma_{p,J_m}^{(m)}$ (or $\Gamma_p^{(1)}$, respectively) is an indecomposable K -representation of degree $m \cdot |H_{p^n}|$ of the group H_{p^n} ($n \geq 2$) (or of degree $|H_{p^2}|$ of the group H_{p^2} for $n = 2$, respectively).*

Proof. Representations which depend on matrix parameters in this way were studied in [1,2]. Using methods and results from these papers, it is not difficult to show for $n > 2$ that the K -representations $\Gamma_{p,A}^{(m)}$ and $\Gamma_{p,B}^{(m)}$ are equivalent if and only if

$$C^{-1}AC - B \equiv 0 \pmod{p}, \quad (6)$$

for some invertible matrix C . Moreover, the K -representation $\Gamma_{p,A}^{(m)}$ is decomposable if and only if there is a decomposable matrix B , which satisfies (6). In particular, the K -representation $\Gamma_{p,J_m}^{(m)}$ is an indecomposable K -representation of H_{p^n} .

The case $n = 2$ follows from [1]. The lemma is proved.

Put

$$\Gamma_p^{(m)} = \begin{cases} \Gamma_{p, J_m}^{(m)} & \text{for } n > 2, m > 1; \\ \Gamma_{p, 1}^{(1)} & \text{for } n > 2, m = 1; \\ \Gamma_p^{(1)} & \text{for } n = 2. \end{cases} \quad (7)$$

Lemma 2. *Let L_p be a KH_{p^n} -module affording the K -representation $\Gamma_p^{(m)}$ of the group H_{p^n} ($n \geq 2$) and v_1, v_2, \dots, v_t be a K -basis corresponding to this representation in L_p . Then Kv_1 is a KH_{p^n} -submodule in L_p , which over K has a direct complement L'_p with the following K -basis $\{w_i = v_i + \lambda_i v_1\}$ for some $\lambda_i \in K$ ($i = 2, \dots, t$) and which is left invariant with respect to the operator a^p .*

Proof. Let $n > 2$. Clearly, $a \cdot v_1 = v_1$, i.e. Kv_1 is a KH_{p^n} -submodule in L_p . Using (2) it is easy to check that in the matrix $\Gamma_p^{(m)}(a^p)$ the intertwining matrix

$$U(a^p) = \sum_{t=0}^{p-1} \Delta_1^{p-t-1}(a) \cdot U(a) \cdot \Delta_2^t(a)$$

has the form

$$U(a^p) = \begin{pmatrix} J_m \otimes U_{11} & \dots & E_m \otimes U_{1, n-1} \\ E_m \otimes U_{21} & \dots & E_m \otimes U_{2, n-1} \end{pmatrix},$$

where

$$U_{1i} = (\langle 1 \rangle_i^0, \dots, \langle 1 \rangle_i^0), \quad U_{2i} = (\langle 1 \rangle_i^1, \langle \xi_1 \rangle_i^1, \dots, \langle \xi_1^{p-1} \rangle_i^1) \quad (i = 1, \dots, n-1).$$

We change the basis elements v_{m+i} by $w_{m+i} = v_{m+i} + v_1$ ($i = 1, \dots, p-1$). Since the sum $-(v_{m+1} + \dots + v_{m+p-1}) + v_1$ is replaced by $-(w_{m+1} + \dots + w_{m+p-1}) + pv_1$, this changes the first row of the matrix $U(a^p)$ turning its elements either to zero or to multiples of p . From (4) (for $i = 0$) provides the possibility of changing the basis elements as

$$w_{m+i} = v_{m+i} + \lambda_i \cdot v_1 \quad (p \leq i, \quad \lambda_i \in K),$$

and so we get a K -module L'_p , such that $L_p = Kv_1 \oplus L'_p$ and L'_p is left invariant with respect to the operator a^p .

For $n = 2$ the statement of the lemma is trivial and the lemma is proved.

For the remainder of this chapter we suppose that $K = \mathbb{Z}$ and we will consider only classical crystallographic groups. Let $G = H_{p_1^{n_1}} \times \dots \times H_{p_s^{n_s}}$ be a decomposition of the cyclic group of order $|G| = p_1^{n_1} \dots p_s^{n_s}$ into the cyclic subgroups $H_{p_i^{n_i}}$ of order

$|H_{p_i}^{n_i}| = p_i^{n_i}$, where p_1, \dots, p_s are pairwise distinct primes, $n_1 \geq 3$ and if $s \geq 2$ then $n_2 \geq 2, \dots, n_s \geq 2$.

Define by $\Gamma^{(m)}$ the tensor product of the \mathbb{Z} -representation $\Gamma_{p_1}^{(m)}$ of the group $H_{p_1}^{n_1}$ and the \mathbb{Z} -representations $\Gamma_{p_j}^{(1)}$ of the groups $H_{p_j}^{n_j}$ ($m \in \mathbb{N}$, $j = 2, \dots, s$). Then $\Gamma^{(m)}$ is a \mathbb{Z} -representation of the group G in which

$$\Gamma^{(m)}(a_1^{t_1}, \dots, a_s^{t_s}) = \Gamma_{p_1}^{(m)}(a_1^{t_1}) \otimes \Gamma_{p_2}^{(1)}(a_2^{t_2}) \otimes \dots \otimes \Gamma_{p_s}^{(1)}(a_s^{t_s}).$$

Lemma 3. *If $(m, |G|) = 1$ then $\Gamma^{(m)}$ is an indecomposable \mathbb{Z} -representation of G .*

Proof. Let $\Gamma^{(m)}|_{H_{p_i}^{n_i}}$ be the restriction of the representation $\Gamma^{(m)}$ to $H_{p_i}^{n_i}$. From Lemma 1 it follows that the degree of each indecomposable summand of $\Gamma^{(m)}|_{H_{p_i}^{n_i}}$ is $m \cdot |H_{p_1}^{n_1}|$ for $i = 1$ and $|H_{p_i}^{n_i}|$ for $i > 1$, respectively.

If $\Gamma^{(m)}$ is decomposable and Γ is a summand in $\Gamma^{(m)}$, then the degree of Γ is divisible by the number $m \cdot |H_{p_1}^{n_1}|$ and if $s \geq 2$ then by the numbers $|H_{p_2}^{n_2}|, \dots, |H_{p_s}^{n_s}|$ (see Lemma 1 for the case $K = \mathbb{Z}_p$).

Thus from the condition $(m, |G|) = 1$ it follows that $\Gamma = \Gamma^{(m)}$. The lemma is proved.

Now we construct a cocycle of the group G . Let M be a $\mathbb{Z}G$ -module of the \mathbb{Z} -representation $\Gamma^{(m)}$ affording the group G and

$$M = L_{p_1} \otimes_K \dots \otimes_K L_{p_s}, \quad (8)$$

where L_{p_i} is a $\mathbb{Z}H_{p_i}^{n_i}$ -submodule for $\Gamma_{p_i}^{(m)}$ ($i = 1, \dots, s$). If $g = a_1^{t_1} \dots a_s^{t_s} \in G$ and $l = l_1 \otimes \dots \otimes l_s \in M$, then

$$g \cdot l = a_1^{t_1} \cdot l_1 \otimes \dots \otimes a_s^{t_s} \cdot l_s,$$

where $l_i \in L_{p_i}$, $t_i \in \mathbb{Z}$ ($i = 1, \dots, s$).

We can suppose that $M \subset \mathbb{R}^d$. Each \mathbb{Z} -basis in M is also an \mathbb{R} -basis in \mathbb{R}^d and an $\mathbb{R}^+/ \mathbb{Z}^+$ -basis in $\widehat{M} = \mathbb{R}^d / M^+$, where $d = m \cdot |G| = \deg(\Gamma^{(m)})$.

Let $v = v_1^{(1)} \otimes \dots \otimes v_s^{(1)}$ be the tensor product of the first \mathbb{Z} -basis elements of the modules L_1, \dots, L_{p_s} . Obviously, $a \cdot v = v$. Define a function $f : G \rightarrow \widehat{M}$ by

$$f(g) = \left(\frac{t_1}{p_1^{n_1}} + \dots + \frac{t_s}{p_s^{n_s}} \right) \cdot v + M, \quad (9)$$

where $g = a_1^{t_1} \dots a_s^{t_s} \in G$, $t_1, \dots, t_s \in \mathbb{Z}$.

Since $g_1 \cdot v = v$ and $f(g_1 \cdot g_2) = f(g_1) + f(g_2)$ for $g_1, g_2 \in G$, we obtain

$$f(g_1 \cdot g_2) = f(g_2) + f(g_1) = g_1 \cdot f(g_2) + f(g_1)$$

and, therefore, f is a 1-cocycle of G in \widehat{M} . The lemma is proved.

Lemma 4. *The restriction of the cocycle f to any prime order subgroup of G is not a coboundary.*

Proof. Let $1 \leq i \leq s$ and put $b = a_i^{p_i^{n_i-1}} \in G$. From Lemma 2 and (8) it follows that the \mathbb{Z} -module M can be decomposed as $M = \mathbb{Z}v \oplus M'$, where $\mathbb{Z}v$ is a $\mathbb{Z}G$ -module, M' is a \mathbb{Z} -module which is left invariant with respect to the operator $a_i^{p_i}$ (moreover with respect to the operator b). Thus $\widehat{M} = Fv \oplus \widehat{M}'$ and $b(\widehat{M}') = \widehat{M}'$. If $z \in \widehat{M}$, then $z = \alpha v + z_1$ ($\alpha \in F, z_1 \in \widehat{M}'$). From (9) and from the condition $b(z_1) \in \widehat{M}'$, it follows that

$$f(b) = \frac{1}{p_i}v + M \neq (b-1)z + M$$

for any $z \in \widehat{M}$. Therefore, the restriction of the cocycle f to $\langle b \rangle$ is not a coboundary which proves the lemma.

Proof of Theorem 1. From Lemma 4 it follows that $\mathfrak{Cnfs}(G; M; T)$ is a torsionfree group. Moreover, according to Lemma 3, $\Gamma^{(m)}(G)$ is an indecomposable subgroup in $GL(d, K)$, where $d = m \cdot |G|$ and $(m, |G|) = 1$. So the proof is complete.

4. PROOF OF THEOREM 2

Let $K = \mathbb{Z}, \mathbb{Z}_{(p)}$ or \mathbb{Z}_p as above and let $\varepsilon = \xi$ be a p^{th} primitive root of unity ($p > 2$). Then $B_1 = \{1, \varepsilon, \dots, \varepsilon^{p-2}\}$ is an \mathfrak{F} -basis in the field $\mathfrak{F}(\varepsilon)$ and a K -basis in the ring $K[\varepsilon]$, where \mathfrak{F} is the field of fractions of the ring K .

The symbol $\langle \alpha \rangle$ denotes a $(p-1)$ -dimensional column, which consists of coordinates of the element $\alpha \in \mathfrak{F}(\varepsilon)$ in the basis B_1 and $\tilde{\alpha}$ denotes the matrix of the operator of multiplication by α in the \mathfrak{F} -basis B_1 of the field $\mathfrak{F}(\varepsilon)$. Clearly, $\tilde{\varepsilon} \cdot \langle \alpha \rangle = \langle \varepsilon \alpha \rangle$.

The group $G = \langle a, b \rangle \cong C_p \times C_p$ ($p > 2$) has the following $p+2$ irreducible K -representations, which are pairwise nonequivalent over the field \mathfrak{F} :

$$\begin{aligned} \gamma_0 : a &\rightarrow 1, \quad b \rightarrow 1; & \gamma_1 : a &\rightarrow \tilde{1}, \quad b \rightarrow \tilde{\varepsilon}; \\ \gamma_2 : a &\rightarrow \tilde{\varepsilon}, \quad b \rightarrow \tilde{1}; & \gamma_3 : a &\rightarrow \tilde{\varepsilon}, \quad b \rightarrow \tilde{\varepsilon}; \\ \rho_i : a &\rightarrow \tilde{\varepsilon}, \quad b \rightarrow \tilde{\varepsilon}^i, \end{aligned} \quad (10)$$

where $i = 2, \dots, p-1$ and $\tilde{1} = E_{p-1}$ is the identity matrix of degree $p-1$.

Put $\tau = \rho_{p-1} \oplus \dots \oplus \rho_2 \oplus \gamma_3 \oplus \gamma_2 \oplus \gamma_1$. Define the following K -representation of the group $G = \langle a, b \rangle$:

$$\Gamma_0 : a \rightarrow \begin{pmatrix} \tau(a) & U(a) \\ 0 & \gamma_0(a) \end{pmatrix}, \quad b \rightarrow \begin{pmatrix} \tau(b) & U(b) \\ 0 & \gamma_0(b) \end{pmatrix};$$

where the intertwining matrix U satisfies:

$$U(a) = \begin{pmatrix} \langle 1 \rangle \\ \vdots \\ \langle 1 \rangle \\ 0 \end{pmatrix}, \quad U(b) = \begin{pmatrix} \langle \alpha_1 \rangle \\ \vdots \\ \langle \alpha_p \rangle \\ \langle 1 \rangle \end{pmatrix};$$

and $\alpha_i = \frac{\varepsilon^{p-i}-1}{\varepsilon-1}$ ($i = 1, 2, \dots, p$).

Lemma 5. Γ_0 is a faithful indecomposable K -representation of $G = \langle a, b \rangle$.

Proof. Using $\tilde{\varepsilon} \cdot \langle \alpha \rangle = \langle \varepsilon \alpha \rangle$ and $1 + \varepsilon + \dots + \varepsilon^{p-1} = 0$ it is easy to see that Γ_0 is a K -representation. Since $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{Z}_p$, it is enough to prove the lemma for $K = \mathbb{Z}_p$. For this it is sufficient to prove the locality of the centralizer of Γ_0 :

$$E(\Gamma_0) = \{ X \in M(p^2, K) \mid X \cdot \Gamma_0(g) = \Gamma_0(g) \cdot X, \quad g \in G \}.$$

Let δ, δ' be representations from (10) and let V be a K -matrix such that

$$\delta(g) \cdot V = V \cdot \delta'(g), \quad (g \in G).$$

Then

$$V = \begin{cases} 0 & \text{if } \delta \neq \delta'; \\ \tilde{x}, \quad \text{where } x \in K[\varepsilon] & \text{if } \delta = \delta' \neq \gamma_0. \end{cases}$$

It follows that $X \in E(\Gamma_0)$ has the following form

$$X = \begin{pmatrix} \tilde{x}_1 & 0 & \cdots & \cdots & 0 & \langle y_1 \rangle \\ & \tilde{x}_2 & 0 & \cdots & 0 & \langle y_2 \rangle \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \tilde{x}_p & 0 & \langle y_p \rangle \\ & & & & \tilde{x}_{p+1} & \langle y_{p+1} \rangle \\ 0 & & & & & x_0 \end{pmatrix},$$

where $x_i = x_0 + (\varepsilon - 1)y_i$, $x_0 \in K$ and $y_i \in K[\varepsilon]$ ($i = 1, 2, \dots, p+1$). From the structure of the matrix X and the condition $K = \mathbb{Z}_p$ we get that X is an invertible matrix if and only if x_0 is a unit in K . Since K is a local ring, then $E(\Gamma_0)$ is also local. The lemma is proved.

Let $M_0 = K^{p^2}$ be the K -module of the K -representation Γ_0 of G consisting of p^2 -dimensional columns over K . It is convenient to further condense the elements of M_0 , considering that the initial $p+1$ coordinates of our vector are $\langle x \mid x \in K[\varepsilon] \rangle$ (i.e. belongs to K^{p-1}), and the final coordinate belong to K . We will do the same with elements of FM_0 (the space of p^2 -dimensional columns over F).

Lemma 6. Let $\alpha = (\varepsilon - 1)^{-1}$ and let X, Y be the following elements from FM_0 :

$$X = \begin{pmatrix} \langle 0 \rangle \\ \vdots \\ \langle 0 \rangle \\ \langle \alpha \rangle \\ 0 \end{pmatrix}; \quad Y = \begin{pmatrix} \langle \alpha \rangle \\ \vdots \\ \langle \alpha \rangle \\ \langle 0 \rangle \\ 0 \end{pmatrix}. \quad (11)$$

There exists a 1-cocycle $f : G = \langle a, b \rangle \cong C_p \times C_p \rightarrow \widehat{M_0} = FM_0^+ / M_0^+$ such that

$$f(a) = X + M_0 \quad \text{and} \quad f(b) = Y + M_0.$$

Moreover, this cocycle f is not cohomologous to the zero cocycle at each nontrivial element of G .

Proof. Note that $\alpha = (\varepsilon - 1)^{-1} \in \mathfrak{F}(\varepsilon)$ does not belong to $K[\varepsilon]$, but $p\alpha \in K[\varepsilon]$. It is easy to see that the initial $p + 1$ diagonal quadratic blocks of the matrix

$$(\Gamma_0^{p-1} + \Gamma_0^{p-2} + \cdots + \Gamma_0 + E_{p^2})(g), \quad (g \in G)$$

are either zero or have the form $p \cdot \tilde{1}$, and that the final 1-dimensional block is equal to p . It follows that

$$\begin{aligned} (\Gamma_0^{p-1}(a) + \Gamma_0^{p-2}(a) + \cdots + \Gamma_0(a) + E_{p^2})X &\in M_0, \\ (\Gamma_0^{p-1}(b) + \Gamma_0^{p-2}(b) + \cdots + \Gamma_0(b) + E_{p^2})Y &\in M_0, \\ (\Gamma_0(a) - E_{p^2})Y - (\Gamma_0(b) - E_{p^2})X &\in M_0. \end{aligned} \tag{12}$$

The third condition follows from the equation $(\tilde{\varepsilon} - \tilde{1})\langle \alpha \rangle = \langle 1 \rangle \in K^{p-1}$.

Define a function $f : G = \langle a, b \rangle \cong C_p \times C_p \rightarrow \widehat{M}_0$ by

$$\begin{aligned} f(1) &= M_0; \\ f(a^i) &= (a^{i-1} + \cdots + a + 1)X + M_0; \\ f(b^j) &= (b^{j-1} + \cdots + b + 1)Y + M_0; \\ f(a^i b^j) &= a^i f(b^j) + f(a^i), \end{aligned} \tag{13}$$

where $i, j = 1, \dots, p-1$.

According to (11)–(13) we get that f is a cocycle of G with value in \widehat{M}_0 . To prove the rest of the statement it is sufficient to consider only the generating elements $\{a, a^i b \mid i = 0, \dots, p-1\}$ of all different nontrivial cyclic subgroups of G .

If $x \in FM_0$, then by $x_{(s)}$ we define the s^{th} condensed coordinate of the vector x ($s = 1, \dots, p+1$). We will do the same with elements of \widehat{M}_0 . Then by (13) we obtain that

$$f_{(s)}(a^s b) = \langle \varepsilon^s \alpha \rangle + K^{p-1} \quad (s = 1, \dots, p), \quad f_{(p+1)}(a) = \langle \alpha \rangle + K^{p-1}. \tag{14}$$

It is easy to see that

$$\Gamma_0(a^s) = \begin{pmatrix} \tilde{\varepsilon}^s & 0 & \cdots & \cdots & 0 & \langle \beta_s \rangle \\ \tilde{\varepsilon}^s & 0 & \cdots & 0 & 0 & \langle \beta_s \rangle \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \tilde{\varepsilon}^s & 0 & \langle \beta_s \rangle \\ 0 & & & & \tilde{1} & 0 \\ & & & & & 1 \end{pmatrix},$$

where $\beta_s = \frac{\varepsilon^s - 1}{\varepsilon - 1}$ ($s = 1, 2, \dots, p$). Since $\varepsilon^s \alpha_s + \beta_s = 0$ ($s = 1, 2, \dots, p$) (see the notation before Lemma 5) we have that $(p-1)$ rows of the matrix $\Gamma_0(a^s b) - E_{p^2}$

corresponding to the s^{th} diagonal block will be zero. Besides, the final p rows of the matrix $\Gamma_0(a^s b) - E_{p^2}$ are also zero. Thus, for any vector $z \in FM_0$ the s^{th} condensed coordinate of the vector $(\Gamma_0(a^s b) - E_{p^2})z$ ($s = 1, 2, \dots, p$) will be equal to zero. The $(p+1)^{th}$ coordinate in $(\Gamma_0(a) - E_{p^2})z$ will also be zero.

Hence, according to (14) and the condition $\alpha = (\varepsilon - 1)^{-1} \notin K[\varepsilon]$ it follows that

$$(\Gamma_0(a^s b) - E_{p^2})z + f(a^s b) \neq M_0 \quad \text{and} \quad (\Gamma_0(a) - E_{p^2})z + f(a) \neq M_0$$

for any $z \in FM_0$ ($s = 1, 2, \dots, p$). The lemma is proved.

Corollary 1. *The group $\mathfrak{Crys}(G; M_0; f)$ is torsionfree.*

Let us define a K -representation of the group $G = \langle a, b \rangle$ as follows. Set

$$\Delta_n = \begin{pmatrix} E_n \otimes \gamma_3 & 0 & u_{11} & u_{12} \\ & E_n \otimes \gamma_2 & u_{21} & u_{22} \\ & & E_n \otimes \gamma_1 & 0 \\ 0 & & & E_n \otimes \gamma_0 \end{pmatrix},$$

where

$$\begin{aligned} u_{11}(a) &= u_{21}(a) = -u_{21}(b) = E_n \otimes \tilde{1}, & u_{11}(b) &= u_{22}(b) = 0, \\ u_{12}(a) &= u_{12}(b) = J_n \otimes \langle 1 \rangle, & u_{22}(a) &= E_n \otimes \langle 1 \rangle, \end{aligned}$$

and J_n is the upper triangular Jordan block of degree n .

Lemma 7. *The K -representation of the group $G = \langle a, b \rangle$ defined by*

$$a \rightarrow \Delta_n(a), \quad b \rightarrow \Delta_n(b)$$

is indecomposable.

Proof. See [1,5].

Using the representation Γ_0 we define the following K -representation of G :

$$\Gamma_n = \begin{pmatrix} \Gamma_0 & V_n \\ 0 & \Delta_n \end{pmatrix},$$

where V_n is the matrix, which elements are intertwining functions of the composition factors in Γ_0 with the composition factors in Δ_n . All these intertwining functions are zero except the function v which intertwines γ_3 in Γ_0 with the first representation γ_0 in $E_n \otimes \gamma_0$ and $v(a) = v(b) = \langle 1 \rangle$. Thus

$$V_n(a) = V_n(b) = \begin{pmatrix} 0 \dots 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 \dots 0 & 0 & 0 & \dots & 0 \\ 0 \dots 0 & \langle 1 \rangle & \langle 0 \rangle & \dots & \langle 0 \rangle \\ 0 \dots 0 & \langle 0 \rangle & \langle 0 \rangle & \dots & \langle 0 \rangle \\ 0 \dots 0 & \langle 0 \rangle & \langle 0 \rangle & \dots & \langle 0 \rangle \\ 0 \dots 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Lemma 8. Γ_n is an indecomposable K -representation of $G = \langle a, b \rangle$.

Proof. Of course Γ_n is a K -representation which is equivalent to the following K -representation of the group G :

$$\Gamma'_n = \begin{pmatrix} \rho_{p-1} + \cdots + \rho_2 & V'_n \\ 0 & \Delta'_{n+1} \end{pmatrix}, \quad (15)$$

where Δ'_{n+1} differs from Δ_{n+1} only by the intertwining matrix $U' = (u'_{ij})$ (the notation for Δ_n , u_{ij} was introduced after Corollary 1, and that for Γ_0 , before Lemma 5):

$$\begin{aligned} u'_{12}(a) &= u'_{12}(b) = J'_{n+1} \otimes \langle 1 \rangle, \\ u'_{22}(a) &= E_{n+1} \otimes \langle 1 \rangle, \\ u'_{11}(b) &= u'_{22}(b) = 0, \\ u'_{11}(a) &= u'_{21}(a) = -u'_{21}(b) = \begin{pmatrix} \tilde{0} & 0 \\ 0 & E_n \otimes \tilde{1} \end{pmatrix}. \end{aligned} \quad (16)$$

Moreover, in the representation Δ'_{n+1} there is a non-zero intertwining between the first γ_1 and the first γ_0 : $u(a) = 0$, $u(b) = \langle 1 \rangle$. Note that we obtained Γ'_n from Γ_n by a permutation of the indecomposable components, where intertwining functions of Γ'_n were obtained from the corresponding ones of Γ_n . If Γ'_n is decomposable, then either the representations $\rho_{p-1}, \dots, \rho_2$ or their sum cannot be components in Γ'_n . Each of these representations has nonzero intertwining with γ_0 , which cannot be changed without changing the zero intertwining for $\rho_{p-1}, \dots, \rho_2$. Thus for the decomposability of the representation Γ'_n the decomposability of the representation Δ'_{n+1} is necessary.

The K -representation Δ'_{n+1} of G is indecomposable. Indeed, the additive group of the intertwining functions for any pairs of different irreducible K -representations (10) of the group G is isomorphic to the additive group of the field $K_p = K/pK$. Any equivalence transformation (over K) acting on Δ'_{n+1} will change the intertwining functions of the different pairs of the irreducible components of Δ'_{n+1} . If we change the intertwining functions by elements of the field K_p then the effect of the functions is replaced by the effect of elements of the field K_p . As a consequence the K -representation Δ'_{n+1} can be parametrized by the following matrix over K_p :

$$C = \begin{pmatrix} E'_n & J'_{n+1} \\ E'_n & E_{n+1} \end{pmatrix},$$

where $E'_n = \begin{pmatrix} 0 & 0 \\ 0 & E_n \end{pmatrix}$ (the notation for Δ_n was introduced before Lemma 7 and in (15)-(16)).

The representation Δ'_{n+1} is decomposable over K if and only if there exist matrices $S_i \in GL(n+1, K_p)$ ($i = 1, \dots, 4$) such that

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}^{-1} \cdot C \cdot \begin{pmatrix} S_3 & 0 \\ 0 & S_4 \end{pmatrix} = \begin{pmatrix} E'_n & X \\ E'_n & E_{n+1} \end{pmatrix}, \quad (17)$$

where

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \quad (18)$$

is decomposable over K_p , and X_1, X_2 are square matrices.

Now, suppose that Δ'_{n+1} is decomposable and satisfies (17)-(18). It follows that

$$S_1 = \begin{pmatrix} t_1^{-1} & 0 \\ * & S \end{pmatrix}, \quad S_2 = S_4 = \begin{pmatrix} t_2 & 0 \\ * & S \end{pmatrix},$$

where $t_1 \cdot t_2 \neq 0$, $S \in GL(n, K_p)$ and

$$X = S_1^{-1} \cdot J'_{n+1} \cdot S_2 = T^{-1} \cdot Y \cdot T, \quad (19)$$

where

$$T = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}, \quad Y = \begin{pmatrix} t_1 \cdot t_2 & y_{12} \\ y_{21} & Y_n \end{pmatrix}. \quad (20)$$

Here Y has the following description: $t_1 \cdot t_2 \neq 0$, $y_{12} = (t_1, 0, \dots, 0)$, y_{21} is an n -dimensional column, Y_n is a matrix which is obtained from the matrix J'_n by changing the first column by an n -dimensional column over K_p . Since $y_{12}S \neq 0$, then the case $X_1 = t_1 t_2$, $X_2 = S^{-1}Y_n S$ is impossible, where X_1, X_2 are defined in (18). Thus

$$S^{-1} \cdot Y_n \cdot S = \begin{pmatrix} * & 0 \\ 0 & X_2 \end{pmatrix}. \quad (21)$$

Let $\overline{K_p}$ be the algebraic closure of the field K_p . The equivalence transformation with T over $\overline{K_p}$ given in (19) can be used to continue the decomposition of the matrix X such that X_2 splits into Jordan blocks over $\overline{K_p}$ (see (17)-(21)).

Of course, we can arrange that X_2 is $J_s(\alpha)$, the Jordan block with α in the main diagonal. The matrix Y we can be considered as a linear operator on the space $\overline{K_p}^{n+1}$ of $n+1$ -dimension column vectors. Thus from (18) it follows that X is the matrix of the operator Y in that basis of the space $\overline{K_p}^{n+1}$, which consist of the columns of the matrix T . The Jordan block $X_2 = J_s(\alpha)$ corresponds to the eigenvector $e \in \overline{K_p}^{n+1}$ of the operator Y : $Ye = \alpha e$. Since X_2 does not include the first column of X , the vector e is a column of the matrix T , which is different from the first column, i.e. the first component of e is equal to zero. Using the description of Y in (19), it is easy to show that the equality $Ye = \alpha e$ ($\alpha \in \overline{K_p}$) is impossible for the vector $e = (0, \gamma_1, \dots, \gamma_n)^T \neq 0$. This contradicts the decomposability of the K -representation Δ'_{n+1} of G . The lemma is proved.

Proof of Theorem 2. We can suppose that $M \subset F^{d_n}$. Each K -basis in M is also an F -basis in F^{d_n} and an (F^+/K^+) -basis in $\widehat{M} = F^{d_n+}/M^+$, where $d_n = \deg(\Gamma_n) = (3p-2)n+p^2$. Thus the dimension of the group $\mathbf{Crns}(G; M_n; T_n)$ is equal to d_n , the degree of the representation Γ_n , and is an unbounded function of n . The theorem is proved.

5. PROOF OF THEOREM 3

In this chapter we suppose that $K = \mathbb{Z}$ and we will consider only classical crystallographic groups. Let $A_4 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$ be the alternating group of degree 4. Using [8], we consider the following \mathbb{Z} -representations of A_4 :

$$\begin{aligned} \Delta_1 : \quad a &\rightarrow 1, & b &\rightarrow 1; & \Delta_2 : \quad a &\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & b &\rightarrow \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}; \\ \Delta_3 : \quad a &\rightarrow \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, & b &\rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; & \Delta_4 : \quad a &\rightarrow \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & b &\rightarrow \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \\ \Delta : \quad a &\rightarrow \Delta(a), & b &\rightarrow \Delta(b); & \Gamma_n : \quad a &\rightarrow \Gamma_n(a), & b &\rightarrow \Gamma_n(b); \end{aligned}$$

where

$$\begin{aligned} \Delta &= \begin{pmatrix} \Delta_3 & 0 & X_1 & X_3 \\ & \Delta_3 & X_2 & 0 \\ & & \Delta_2 & 0 \\ 0 & & & \Delta_4 \end{pmatrix}, & \Gamma_n &= \begin{pmatrix} E_n \otimes \Delta_1 & U \\ 0 & E_n \otimes \Delta \end{pmatrix}, \\ X_1(a) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}, & X_2(a) &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ -1 & 0 \end{pmatrix}, & X_3(a) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\ X_i(b) &= 0, & i &= 1, 2, 3, \end{aligned}$$

$$U(a) = E_n \otimes \alpha + J_n(0) \otimes \beta, \quad U(b) = 0,$$

$$\alpha = (0, 0, 0, 0, 2, 0, 1, -1, 0, 0, 0), \quad \beta = (0, -2, 0, 0, 0, 0, 0, 1, -1, -1, 0),$$

$J_n(\nu)$ is the Jordan block of degree n with ν in the main diagonal.

As was proved in [8], the representations $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 are irreducible and Δ and Γ_n are indecomposable \mathbb{Z} -representations.

Let M_n be a \mathbb{Z} -module affording the representation Γ_n of A_4 consisting of d_n dimensional columns over \mathbb{Z} , where

$$\deg(\Gamma_n) = d_n = 12n.$$

It is easy to check that $f_n : A_4 \rightarrow \widehat{M_n}$ defined by

$$f_n(a) = (\underbrace{0, \dots, 0}_{n+3}, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0)^T + M_n; \quad f_n(b) = (\frac{1}{3}, 0, \dots, 0)^T + M_n$$

is a 1-cocycle, which is special. Therefore, we obtain

Corollary 2. *The classical crystallographic group $\mathfrak{Crys}(A_4; M_n; f_n)$ is torsionfree.*

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